Exercice 1 *Solve the following linear system in $\mathbb{Z}/_{5\mathbb{Z}}$ using Gaussian elimination :*

$$\begin{cases} x + 2y + 2z = 3 \\ 2x + z = 4 \\ 3x + y + 3z = 1 \end{cases}$$

Exercice 2 .

1. Let $\mathcal{C} \subset \mathcal{A}^n$ be a code of length n and a minimum distance d show that $|\mathcal{C}| \leq |\mathcal{A}|^{n-d+1}$

2. Deduce that if $C \subset \mathcal{A}^n$ is a linear code of dimension k that

$$k + d \leq n + 1$$

This result is known as the **Singleton bound theorem**

Exercice 3 *Show that in a binary linear code , either all the codewords have even Hamming weights or else the number of odd-weight and even-weight codewords are equal.*

Exercice 4 .

1. Let H be a subgroup of a group G. For any element $g \in G$, there is a bijection between H and $g + H$

2. Let G be a group with a finite number of elements and let H be a subgroup of G. Then the number r of distinct left cosets of H is equal to $|G| / |H|$. In particular both $|H|$ and r divide $|G|$.

Exercice 5 *Let H be the parity check matrix of a linear code C that has both odd and even weight codewords. Construct a new linear code C_1 with the following parity-check matrix*

$$H_1 = \left(\begin{array}{c|cccc} 0 & & & & \\ 0 & & & & \\ \ldots & & & H & \\ 0 & & & & \\ 1 & 1 & 1 & \ldots & 1 \end{array} \right)$$

1. Show that every codeword of C_1 has even weight

2. Show that C_1 can be obtained from C by adding an extra parity check digit denoted by x_1 where

$$x_1 = \begin{cases} 1 \text{ if } w\left(x_2 \ldots x_n, x_{n+1}\right) \text{ is odd} \\ 0 \text{ if } w\left(x_2 \ldots x_n, x_{n+1}\right) \text{ is even} \end{cases}$$

Exercice 6 *The International Standard Book Number (ISBN) is a numeric book identifier used for the ease of handling books particularly by booksellers and libraries.*
According to the 2001 standard, a unique 10-digit identifier is assigned to each book (based on the language of the publishing country, publisher, and the title) and a check digit is then affixed to the identifier. The aim of the checksum is to facilitate detection of two common typing errors made in book handling: Typing a wrong digit and interchanging two subsequent digits.
Taking the check digit into account, a valid ISBN can be regarded as a vector $x = (x_1, \ldots, x_{10})$
where $x_2, \ldots, x_{10} \in \{0, \ldots, 9\}$ *and* $x_1 \in \{0, \ldots, 9, 10\}$ *is the checksum, computed according to the rule :*

$$\sum_{i=1}^{10} i x_i = 0 \bmod 11$$

1. Show that the ISBN code can detect a single error.
2. Show that it can detect transposition of any digit with an adjacent digit.
3. What is the minimum distance of this code?
4. If we used the simpler rule

$$\sum_{i=1}^{10} i x_i = 0 \bmod 10$$

instead of the one above, could the code still detect errors? What about transpositions ?